The Born Rule Dies

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Based on

"Typicality Derived," Phys. Rev. D78, 023514 (2008); arXiv: 0804.3592.

"Cosmological Measures without Volume Weighting," JCAP 0810:025 (2008); arXiv:0808.0351.

"Insufficiency of the Quantum State for Deducing Observational Probabilities," Phys. Lett. B, in press; arXiv: 0808.0722.

"The Born Rule Dies," arXiv: 0903.4888.

Elements of a Complete Theory of the Universe

1. Physical variables (arguments of ψ?)
2. Dynamical laws ('TOE')
3. Boundary conditions (quantum state)
4. Probability rules (analogue of Born's rule)
5. What has probabilities?
6. What do the probabilities mean?

Want complete theories T_i predicting normalized probabilities $P_j(i)$ of observations O_j, $P_j(i) \equiv P(O_j|T_i)$, $\sum_j P_j(i) = 1$.

The Born Rule

Traditional quantum theory uses Born's rule:

$P_j(i) \equiv P(O_j|T_i) = \langle \mathbb{P}_j \rangle_i$,

$\mathbb{P}_j$ = projection operator for observation j,

$\langle\ \rangle_i$ = expectation value of operator inside in the quantum state given by theory T_i.

E.g., $\langle \mathbb{P}_j \rangle_i = \langle i|\mathbb{P}_j|i\rangle$ in pure state $|i\rangle$.

Born's rule seems to work well in ordinary single laboratory settings, where the observer can distinguish between a set of mutually exclusive outcomes.

However, it does not work in a universe large enough that the same observation may occur at different locations, with no observer within the universe (a frog) having access to both observations.

Bird = superobserver who can see the entire universe

Frog = localized observer who can see only one part.

The Problem with Born's Rule

Suppose spacetime has N regions labeled by L, $1 \le L \le N$, but suppose local observations by frogs do not determine L.

Suppose $\mathbb{P}_j^L \mathbb{P}_k^L = \delta_{jk} \mathbb{P}_j^L$ but $\mathbb{P}_j^L \mathbb{P}_k^M \neq 0$ for $j \neq k$, $L \neq M$.

One cannot say $P_j(i) = \langle \mathbb{P}_j^L \rangle_i$, since L isn't known.

One could try $P_j(i) = \langle \mathbb{P}_j \rangle_i$ with $\mathbb{P}_j = I - \prod_L (I - \mathbb{P}_j^L)$.

But then $\sum_j P_j(i) > 1$, since O_j can occur in one region and O_k in another.

$\therefore$ The Born rule does not work when there may be copies of the observer.

The Born rule dies.

Toy Model Showing Death of Born's Rule

Let $|\psi\rangle = \sum_{N=1}^{\infty} a_N |\psi_N\rangle$, where each $|\psi_N\rangle$ has N regions or sites where 0 or 1 observation occurs. (Different N model different sizes of the universe.)

Here, let $|\psi\rangle = |\psi_2\rangle = b_{12}|12\rangle + b_{21}|21\rangle$.

$|12\rangle$: $L=1$ has O_1, $L=2$ has O_2.

$|21\rangle$: $L=1$ has O_2, $L=2$ has O_1.

Since each component has the same number of O_1's as O_2's (one of each), expect $P_1 = P_2 = \frac{1}{2}$. But there are not projection operators $\mathbb{P}_1$ and $\mathbb{P}_2$ in this subspace giving $P_j = \langle\psi|\mathbb{P}_j|\psi\rangle = \frac{1}{2}$ for all such $|\psi\rangle$.

E.g., suppose $\mathbb{P}_1 = |\psi_{12}\rangle\langle\psi_{12}|$, $\mathbb{P}_2 = |\psi'_{12}\rangle\langle\psi'_{12}|$,

$|\psi_{12}\rangle = \cos\theta\,|12\rangle + \sin\theta\, e^{i\phi}|21\rangle$, $|\psi'_{12}\rangle = -\sin\theta\, e^{-i\phi}|12\rangle + \cos\theta|21\rangle$.

Then if $b_{12} = \cos\theta$, $b_{21} = \sin\theta\, e^{i\phi}$, so $|\psi\rangle = |\psi_{12}\rangle$,

$\langle\psi|\mathbb{P}_1|\psi\rangle = \langle\psi_{12}|\psi_{12}\rangle\langle\psi_{12}|\psi_{12}\rangle = 1 \neq \frac{1}{2}$ and

$\langle\psi|\mathbb{P}_2|\psi\rangle = \langle\psi_{12}|\psi'_{12}\rangle\langle\psi'_{12}|\psi_{12}\rangle = 0 \neq \frac{1}{2}$.

Replacements of Born's Rule

Principles we might use as guides:

1. No Extra Vision Principle (NEVP):

   The probabilities of observations that have zero amplitudes to occur anywhere in the quantum state are zero; one cannot see what is not there in the quantum state.

2. Probability Symmetric Principle (PSP):

   If the quantum state is an eigenstate of equal numbers of observations of two different observations, then the probabilities of these two observations are equal.

3. Probability Fraction Principle (PFP):

   If the quantum state is an eigenstate of the fraction of the number of each possible observation to the total number of all possible observations, then the probability of each observation is that fraction.

Plausible Rule for Fixed N

If $|\psi\rangle = |\psi_N\rangle$ with a fixed number of sites N, say $P_j = \frac{p_j}{\sum_k p_k}$ with relative probabilities $p_j = \langle N_j \rangle_1 = \langle \sum_L \mathbb{P}_j^L$ the expectation value of the number of O_j observation

For $N = 2$, $|\psi\rangle = |\psi_2\rangle = b_{11}|11\rangle + b_{12}|12\rangle + b_{21}|21\rangle + b_{22}|22\rangle$,

$\mathbb{P}_1^1 = |11\rangle\langle 11| + |12\rangle\langle 12|$, $\mathbb{P}_1^2 = |11\rangle\langle 11| + |21\rangle\langle 21|$,

$\mathbb{P}_2^1 = |21\rangle\langle 21| + |22\rangle\langle 22|$, $\mathbb{P}_2^2 = |12\rangle\langle 12| + |22\rangle\langle 22|$.

$N_1 = \mathbb{P}_1^1 + \mathbb{P}_1^2 = 2|11\rangle\langle 11| + |12\rangle\langle 12| + |21\rangle\langle 21|$,

$N_2 = \mathbb{P}_2^1 + \mathbb{P}_2^2 = \qquad |12\rangle\langle 12| + |21\rangle\langle 21| + 2|22\rangle\langle 22|$.

$p_1 = \langle\psi|N_1|\psi\rangle = 2|b_{11}|^2 + |b_{12}|^2 + |b_{21}|^2$,

$p_2 = \langle\psi|N_2|\psi\rangle = \qquad |b_{12}|^2 + |b_{21}|^2 + 2|b_{22}|^2$.

For $\langle\psi|\psi\rangle = 1$, $\sum_k p_k = 2|b_{11}|^2 + 2|b_{12}|^2 + 2|b_{21}|^2 + 2|b_{22}|^2 = 2$, so

$P_1 = |b_{11}|^2 + \frac{1}{2}|b_{12}|^2 + \frac{1}{2}|b_{21}|^2$, $P_2 = \frac{1}{2}|b_{12}|^2 + \frac{1}{2}|b_{21}|^2 + |b_{22}|^2$.

If $|\psi\rangle = b_{12}|12\rangle + b_{21}|21\rangle$, indeed $P_1 = P_2 = \frac{1}{2}$.

However, $P_j \neq \langle\psi|\mathbb{P}_j|\psi\rangle$ for any projection operators $\mathbb{P}_j$ fixed independent of the quantum state, so this is not Born's rule.

What Rule for Variable N?

3. 'Volume weighting': $P_j(i) = \dfrac{p_j(i)}{\sum_k p_k(i)}$, $p_j(3) = \langle N_j \rangle = \langle \sum_L \mathbb{P}_j^L \rangle$

$N_j = |j\rangle\langle j| + 2|jj\rangle\langle jj| + |j\ell\rangle\langle j\ell| + |\ell j\rangle\langle \ell j| + 3|jjj\rangle\langle jjj| + \cdots$

4. 'Volume averaging': $p_j(4) = \langle F_j \rangle$, $F_j = \sum_N \frac{1}{N} N_i = \sum_N \frac{1}{N} \sum_L \mathbb{P}_j^L$,

$$F_j = \frac{1}{1}|j\rangle\langle j| + \frac{1}{2}\left[2|jj\rangle\langle jj| + |j\ell\rangle\langle j\ell| + |\ell j\rangle\langle \ell j|\right]$$
$$+ \frac{1}{3}\left[3|jjj\rangle\langle jjj| + 2|jj\ell\rangle\langle jj\ell| + \cdots\right] + \frac{1}{4}[\cdots] + \cdots$$

F_j is the fraction operator for what fraction of the N sites has the observation O_j, summed over N.

5. 'Observational averaging': $p_j(5) = \langle f_j \rangle$, $f_j = \sum_N \frac{1}{N_o} N_j$, where $N_o = \sum_j N_j$ is the total number operator for all observations in the state component with N sites.

In these three cases, $P_j(i) = \langle Q_j(i) \rangle$, with

$Q_j(i) = \dfrac{q_j(i)}{\langle \sum_k q_k(i) \rangle}$ normalized from unnormalized $q_j(i)$:

$$q_j(3) = \sum_L \mathbb{P}_j^L,$$

$$q_j(4) = \sum_N \frac{1}{N} \sum_L \mathbb{P}_N \mathbb{P}_j^L \mathbb{P}_N,$$

$$q_j(5) = \sum_N \sum_{N_o} \sum_L \frac{1}{N_o} \mathbb{P}_{NN_o} \mathbb{P}_j^L \mathbb{P}_{NN_o},$$

$\mathbb{P}_N = |\psi_N\rangle\langle\psi_N|$ onto N sites, $\mathbb{P}_{NN_o}$ onto N sites with N_o observations.

Results for a Toy Quantum State

$$|\psi\rangle = \cos\theta\,|m_0; m_1; m_2\rangle + \sin\theta\,|n_0; n_1; n_2\rangle$$
$$= \frac{1}{\sqrt{2}}|10^{84}; 10^{10}; 0\rangle + \frac{1}{\sqrt{2}}|10^{10^{56}}; 0; 10^{10^{56}-10^{42}}\rangle.$$

m_0 or n_0: number of sites with no observation.
m_1 or n_1: number of sites with human observation O_1.
m_2 or n_2: number of sites with Boltzmann brain " O_2.

'Volume weighting':

$$P_1(3) = \frac{m_1\cos^2\theta + n_1\sin^2\theta}{(m_1+m_2)\cos^2\theta + (n_1+n_2)\sin^2\theta} \sim 10^{-(10^{56}-10^{42}-10)}$$

'Volume averaging':

$$P_1(4) = \frac{\frac{m_1\cos^2\theta}{m_0+m_1+m_2} + \frac{n_1\sin^2\theta}{n_0+n_1+n_2}}{\frac{(m_1+m_2)\cos^2\theta}{m_0+m_1+m_2} + \frac{(n_1+n_2)\sin^2\theta}{n_0+n_1+n_2}} \sim 1$$

'Observational averaging':

$$P_1(5) = \frac{m_1\cos^2\theta}{m_1+m_2} + \frac{n_1\sin^2\theta}{n_1+n_2} = \frac{1}{2}$$

Born's Rule for Local States

Born's rule does not work for the global quantum state of the universe, $\langle\ \rangle_i$: one cannot find state-independent projection operators $\mathbb{P}_j$ such that $P_j(i) = \langle \mathbb{P}_j \rangle_i$ satisfies reasonable properties such as the Probability Symmetry Principle.

However, one may be able to find a local density matrix $\rho(i)$ for a single site so that $P_j(i) = tr(\mathbb{P}_j \rho(i))$ for some local $\mathbb{P}_j$. If the site is L, one could take $\mathbb{P}_j$ to be the restriction of $\mathbb{P}_j^L$ to the Hilbert space of that site.

Then the ambiguity in the replacements of Born's rule would be the ambiguity of how to calculate $\rho(i)$ from the global quantum state. Say $\rho(i) = r(i)/tr\, r(i)$ with unnormalized Hermitian $r(i)$ with nonnegative eigenvalues.

Say $|\psi\rangle_i = \sum_{N=1}^{\infty} \sum_{N_o=1}^{N} a_{NN_o} |\psi_{NN_o}\rangle_i$; $|\psi_{NN_o}\rangle_i$: N_o obs., N sites.

Let $\rho^L_{NN_o}(i) = \underset{K \neq L}{tr}(|\psi_{NN_o}\rangle_i \langle \psi_{NN_o}|)$.

'Volume weighting': $r(3) = \sum_{N=1}^{\infty} \sum_{N_o=1}^{N} |a_{NN_o}|^2 \sum_{L=1}^{N} \rho^L_{NN_o}(3)$

'Volume averaging': $r(4) = \sum_{N=1}^{\infty} \sum_{N_o=1}^{N} |a_{NN_o}|^2 \frac{1}{N} \sum_{L=1}^{N} \rho^L_{NN_o}(4)$

'Observational averaging': $r(5) = \sum_{N=1}^{\infty} \sum_{N_o=1}^{N} |a_{NN_o}|^2 \frac{1}{N_o} \sum_{L=1}^{N} \rho^L_{NN_o}(5)$

Nonlinear Rules for Probabilities

So far I have discussed linear rules

$$P_j(i) \equiv P(O_j|T_i) = \frac{p_j(i)}{\sum_k p_k(i)} = \langle Q_j(i)\rangle_i = \frac{\langle q_j(i)\rangle}{\sum_k \langle q_k(i)\rangle_i}.$$

(Actually, they aren't quite linear because of the denominator, but the unnormalized relative probabilities $p_j(i) = \langle q_j(i)\rangle_i$ are.)

However, one can also propose nonlinear rules.

'Everett existence probability':

$p_j(1) = 0$ if $\langle \mathbb{P}_j^L \rangle = 0 \ \forall L$,

$p_j(1) = 1$ if $\langle \mathbb{P}_j^L \rangle > 0$ for some L

(nonzero amplitude for the observation to exist).

'Nonlinear bird existence probability':

$p_j(6) = p_j(2)^c$ with real positive $c \neq 1$, where

$p_j(2) = \langle \mathbb{P}_j \rangle = \langle \psi | I - \prod_L (I - \mathbb{P}_j^L) | \psi \rangle$

is the linear quantum probability for a bird to see the existence of at least one observation O_j.

'Nonlinear volume weighting': $p_j(7) = p_j(3)^c = \langle \sum_L \mathbb{P}_j^L \rangle^c$.

'Nonlinear volume averaging': $p_j(8) = p_j(4)^c = \langle \sum_L \sum_N \frac{1}{N} \mathbb{P}_N \mathbb{P}_j^L \mathbb{P}_N \rangle^c$.

'Nonlinear observational averaging': $p_j(9) = p_j(5)^c = \langle \sum_L \sum_{N_o} \sum_N \frac{1}{N_o} \mathbb{P}_{NN_o} \mathbb{P}_j^L \mathbb{P}_{NN_o} \rangle^c$

All of these obey the No Extra Vision Principle and the Probability Symmetry Principle but not the Probability Fraction Principle.

Implications for the Measure Problem

Hartle-Hawking and others: Dynamical 'Theory of Everything' isn't; one also needs the quantum state.

This work: Quantum state is also not enough; one needs new rules for extracting probabilities.

This implies the measure problem in cosmology is more serious than might have been thought: it can't be solved just by knowing the quantum state.

'Volume weighting', 'volume averaging', and 'observational averaging' seem the simplest rules so far. 'Volume weighting' and 'observational averaging' tend to lead to a late-time Boltzmann brain problem. 'Volume averaging' can ameliorate this, though it still may require universe lifetime $\lesssim 10^{10^{42}}$.

One also needs a good quantum state. The Hartle-Hawking no-boundary proposal seems to have an early-time Boltzmann brain problem. Perhaps a symmetric-bounce quantum state is better (work in progress).

In any case, Born's rule does not work in cosmology when the universe is sufficiently large to have many copies of an observation.