Black Holes and Hidden Symmetries

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Why Large Extra Dimensions?

Higher Dimensions: Motivations for Study

Higher Dimensions:

Kaluza-Klein models and Unification;

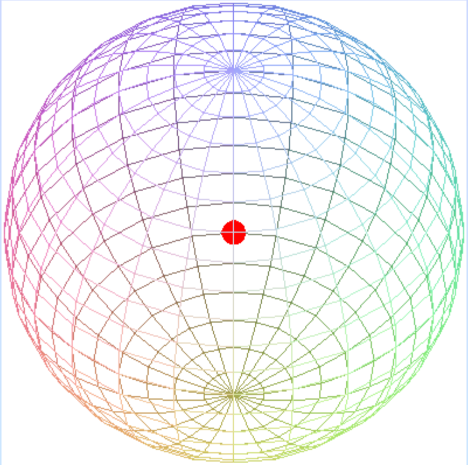
String Theory;

Brane worlds;

`From above view' on Einstein gravity.

Gravity in Higher Dim. Spacetime

$$\Delta^{(4+k)}\Phi = -a_k G^{(k)} M \delta^{3+k}(\vec{x})$$



$$\vec{F} = -\nabla\Phi$$

$$F \sim \frac{G^{(k)} M}{r^{2+k}}$$

No bounded orbits

Gravity at small scales is stronger than in 4D

`Running coupling constant’

$$G(r) = \frac{G^{(k)}}{r^k}, \quad G = G(l), \quad G(r) = G\,\frac{l^k}{r^k}$$

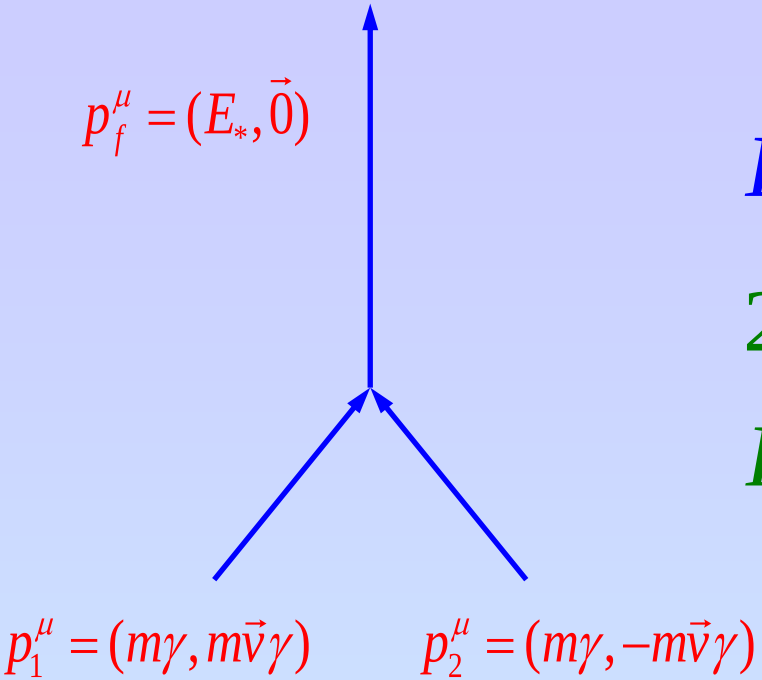
$$\frac{G(r_*)m_p^2}{r_*^2} = \frac{e^2}{r_*^2} \quad \Rightarrow \quad r_* = \left(\frac{Gm_p^2}{e^2}\right)^{1/k} l$$

$$\left(\frac{Gm_p^2}{e^2}\right) \sim 10^{-36}$$

`Hierarchy problem’

$$For\ l \sim 0.01cm\ and\ k = 2 \quad r_* \sim 10^{-20}\,cm$$

$$E = -p_1^{\mu} p_{2\mu} / m, \quad E_* = 2m\gamma, \quad E_* \sim (2Em)^{1/2}$$



$$E = m\gamma^2(1+v^2) \sim 2m\gamma^2 = \gamma E_*,$$

$$2mE \sim E_*^{2},$$

For $E \sim \hbar / r_* \quad E_* \sim 1TeV$

New fundamental scale

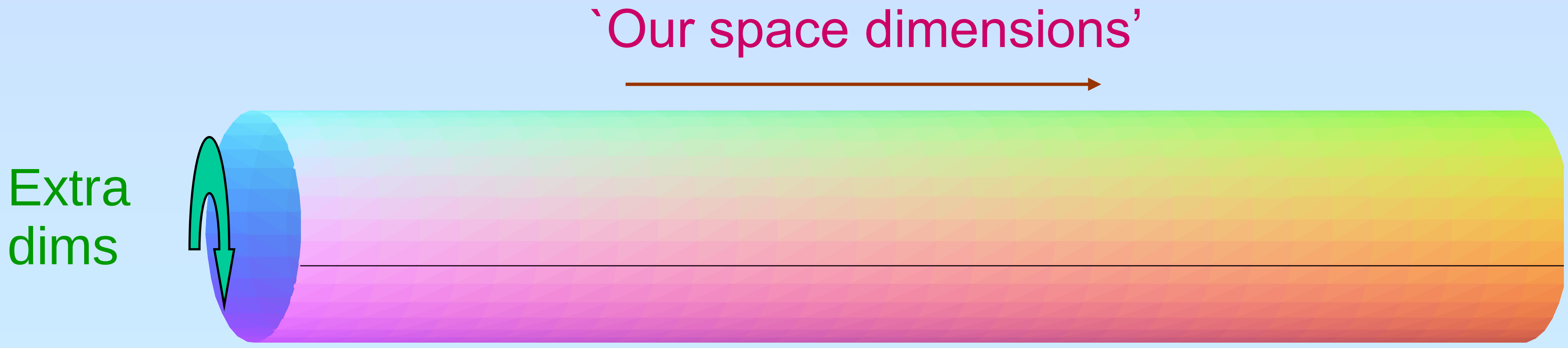
$$M_* = M_{Pl}(l_{Pl}/l)^{\frac{k}{k+2}},$$

For $k = 2$ and $l \sim 0.01cm \quad M_* \sim 1TeV$

4D Newton law is confirmed for r>l.

Q.: How to make gravity strong (HD) at small scales without modifying it at large scales?

A.: Compactification of extra dimensions



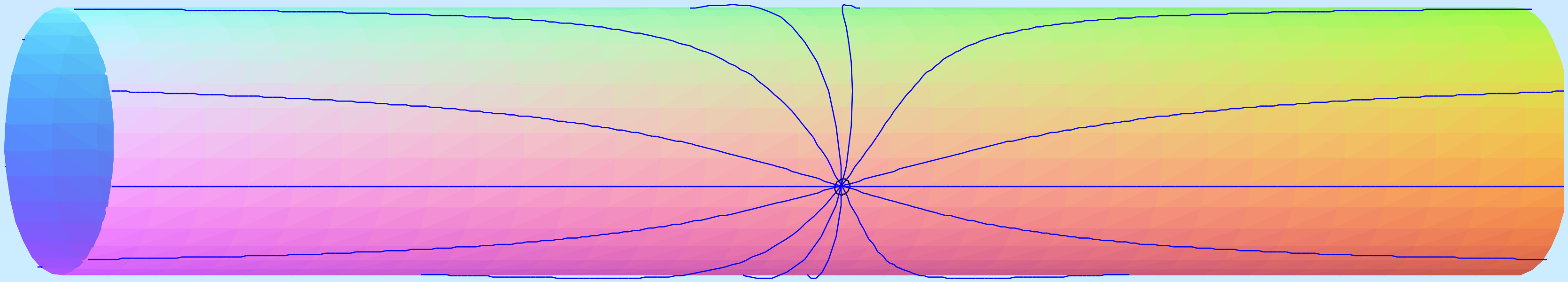
Gravity in ST with Compact Dims

Example: $\Delta^{(4)}\Phi = 0, \quad \mathrm{M}^4 = R^3 \times S^1$

$$\Phi \sim G^* M \sum_{n=-\infty}^{\infty} \frac{1}{\vec{r}^{\,2} + (z+nl)^2} = \frac{G^* M \pi}{lr} \frac{\sinh(\pi r/l)\cosh(\pi r/l)}{\cosh^2(\pi r/l) - \cos^2(\pi z/l)}$$

$$\Phi(\vec{r}, z=0) \sim \frac{G^* M \pi}{lr} \coth(\pi r/l), \quad G = G^* \pi / l$$

$$\mathrm{W} = -\tfrac{1}{2} \sum_n \int dx^4 \left[\nabla_\mu \Phi_n \nabla^\mu \Phi^* + (m_0^2 + \frac{n^2}{L^2}) \Phi \Phi^* \right]$$



Brane World Paradigm

Bosons, fermions and gauge fields are localized within the 4D brane

Gravity is not localized and `lives' in (4+k)-D bulk space

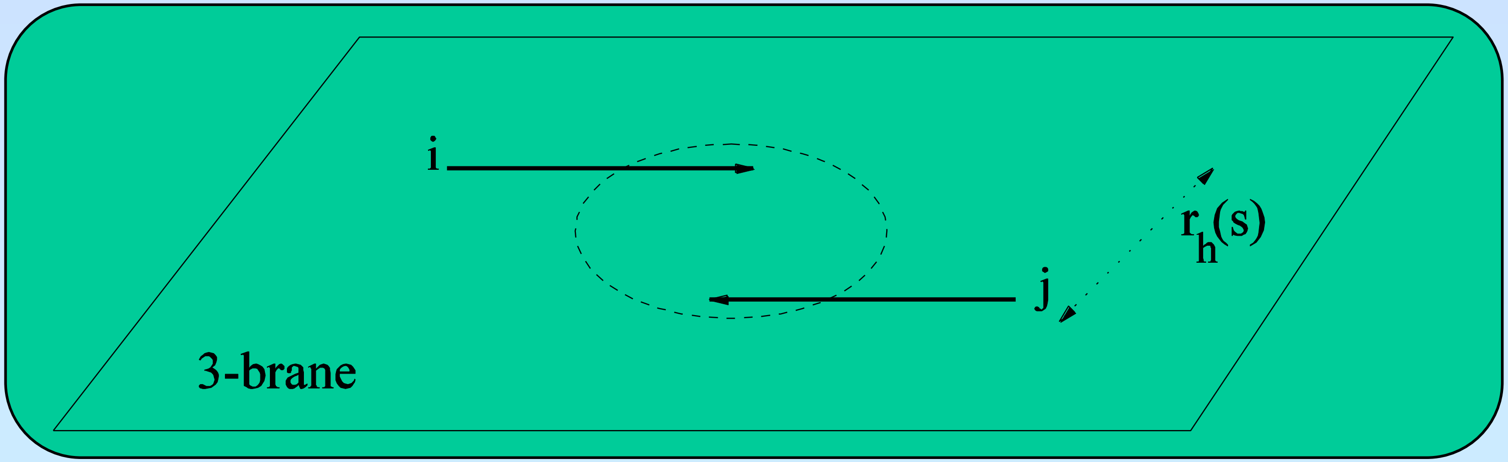
Fundamental scale of order of TeV. Large extra dimensions generate Planckian scales in 4D space

Black Holes as Probes of Extra Dims

We consider first black holes in the mass range

$$10^{-21}g \ll M \ll 10^{26}g$$

Mini BHs creation in colliders

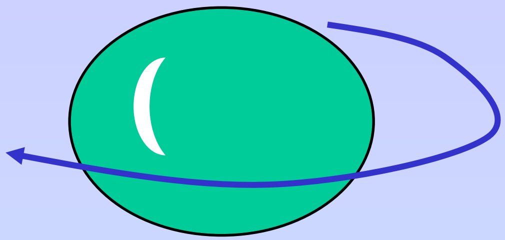


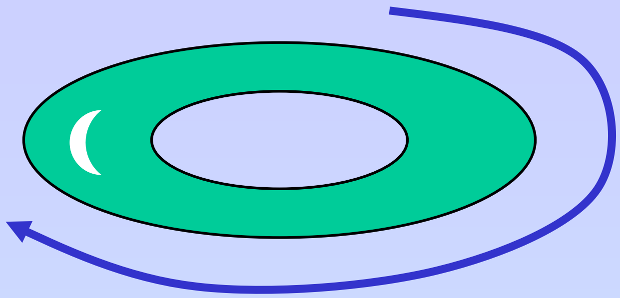
BH formation

Bolding Phase

Thermal (Hawking) decay

Black Objects





Black Holes
[Horizon Topology S^(D-2)]

Black Rings

Black Saturns

Black Strings, ets

Exist in any # of dims

Generic solution is Kerr-NUT-(A)dS

Principal Killing-Yano existence is their characteristic property

Hidden Symmetries, Integrability properties

Higher Dimensional Kerr-NUT-(A)dS Black Holes

Based on:

V. F., D.Kubiznak, Phys.Rev.Lett. 98, 011101 (2007); gr-qc/0605058

D. Kubiznak, V. F., Class.Quant.Grav.24:F1-F6 (2007); gr-qc/0610144

P. Krtous, D. Kubiznak, D. N. Page, V. F., JHEP 0702:004 (2007); hep-th/0612029

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007); hep-th/0611245

D. Kubiznak, V. F., JHEP 0802:007 (2008); arXiv:0711.2300

V. F., Prog. Theor. Phys. Suppl. 172, 210 (2008); arXiv:0712.4157

V.F., David Kubiznak, CQG, 25, 154005 (2008); arXiv:0802.0322

P. Connell, V. F., D. Kubiznak, PRD, 78, 024042 (2008); arXiv:0803.3259

P. Krtous, V. F., D. Kubiznak, PRD 78, 064022 (2008); arXiv:0804.4705

D. Kubiznak, V. F., P. Connell, P. Krtous ,PRD, 79, 024018 (2009); arXiv:0811.0012

D. N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, Phys.Rev.Lett. (2007); hep-th/0611083

P. Krtous, D. Kubiznak, D. N. Page, M. Vasudevan, PRD76:084034 (2007); arXiv:0707.0001

`Alberta Separatists'

Recent Reviews on HD BHs

Panagiota Kanti, “Black holes in theories with large extra dimensions: A Review”, Int.J.Mod.Phys.A19:4899-4951 (2004).

Panagiota Kanti “Black Holes at the LHC”, Lectures given at 4th Aegean Summer School: Black Holes, Mytilene, Island of Lesvos, Greece, (2007).

Roberto Emparan, Harvey S. Reall, “Black Holes in Higher Dimensions” (2008) 76pp, Living Rev.Rel. arXiv:0801.3471

V.F. and David Kubiznak, “Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables”, CQG, Peyresq-Physics 12 workshop, Special Issue (2008) ; arXiv:0802.0322

V.F, “Hidden Symmetries and Black Holes”, NEB-13, Greece, (2008); arXiv:0901.1472

Hidden Symmetries of 4D BHs

Hidden symmetries play an important role in study 4D rotating black holes. They are responsible for separation of variables in the Hamilton-Jacobi, Klein-Gordon and higher spin equations.

Separation of variables allows one to reduce a physical problem to a simpler one in which physical quantities depend on less number of variables. In case of complete separability original partial differential equations reduce to a set of ordinary differential equations

Separation of variables in the Kerr metric is used for study:

(1) Black hole stability
(2) Particle and field propagation
(3) Quasinormal modes
(4) Hawking radiation

Brief History of 4D BHs

1968: Forth integral of motion, separability of the Hamilton-Jacobi and Klein-Gordon equations in the Kerr ST, Carter's family of solutions [Carter, 1968 a, b,c]

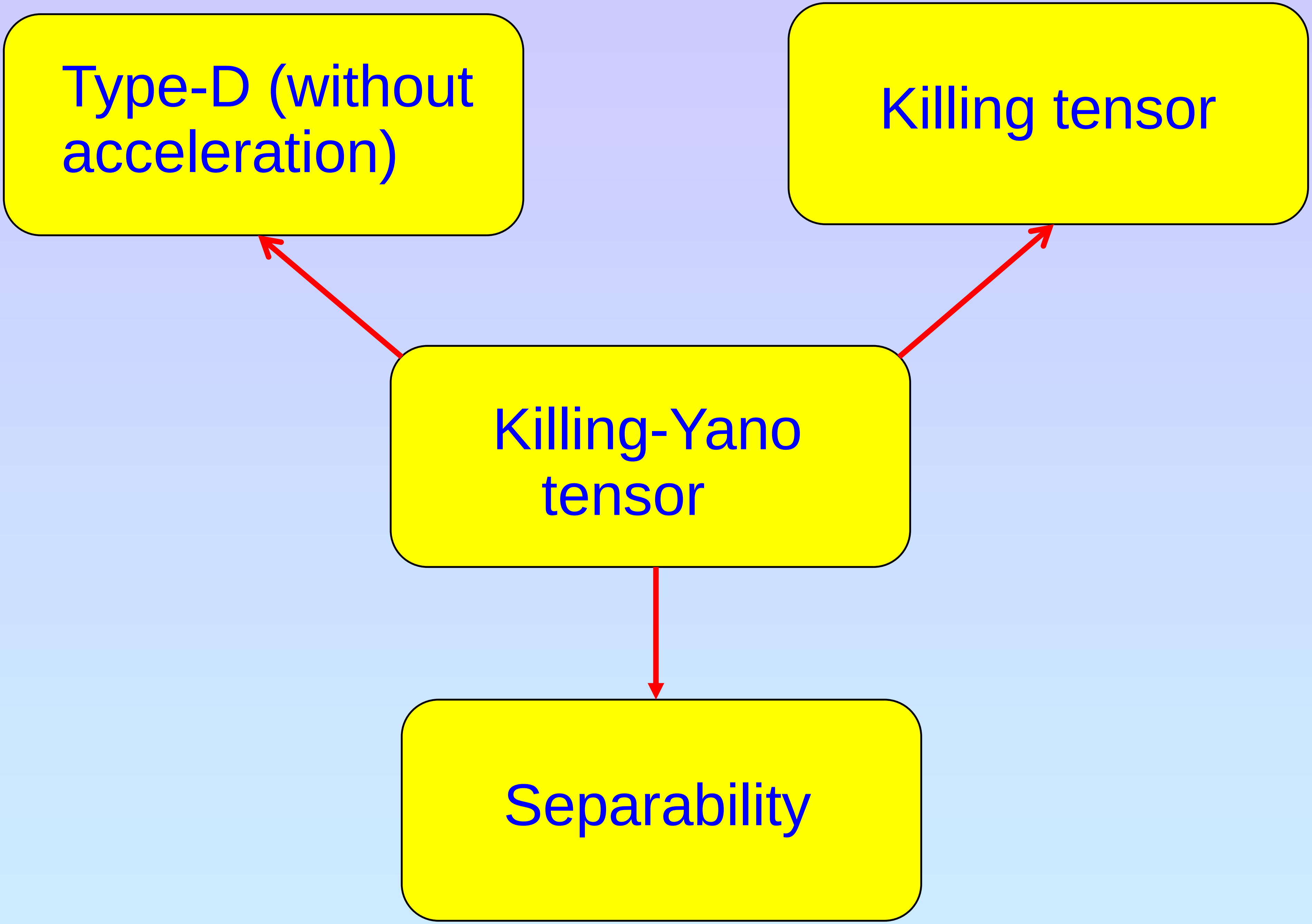
1970: Walker and Penrose pointed out that quadratic in momentum Carter's constant is connected with the symmetric rank 2 Killing tensor

1972: Decoupling and separation of variables in EM and GP equations [Teukolsky]. Massless neutrino case [Teukolsky (1973), Unruh (1973)]. Massive Dirac case [Chandrasekhar (1976), Page (1976)]

1973: Killing tensor is a `square' of antisymmetric rank 2 Killing-Yano tensor [Penrose and Floyd (1973)]

1974: Integrability condition for a non-degenerate Killing-Yano tensor imply that the ST is of Petrov type D [Collinson (1974)]

1975: Killing-Yano tensor generates both symmetries of the Kerr ST [Hughston and Sommers (1975)]

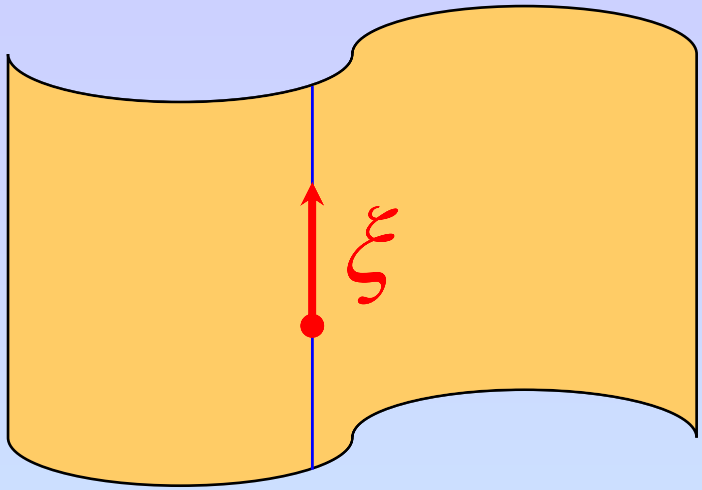

Type-D (without acceleration)
Killing tensor
Killing-Yano tensor
Separability

Main Results (HD BHs)

Rotating black holes in higher dimensions, described by the Kerr-NUT-(A)dS metric, in many aspects are very similar to the 4D Kerr black holes.

- They admit a principal conformal Killing-Yano tensor.
- This tensor generates a tower of Killing tensors and Killing vectors, which are responsible for hidden and `explicit' symmetries.
- The corresponding integrals of motion are sufficient for a complete integrability of geodesic equations.
- These tensors imply separation of variables in Hamilton-Jacoby, Klein-Gordon, and Dirac equations.
- Any solution of the Einstein equations which admits a non-degenerate a principal conformal Killing-Yano tensor is a Kerr-NUT-(A)dS spacetime.

Spacetime Symmetries



$$L_\xi g_{\mu\nu} = 0$$

$$L_\xi g_{\mu\nu} = \alpha g_{\mu\nu}$$

$$\xi_{(\mu;\nu)} = 0 \quad \text{(Killing equation)}$$

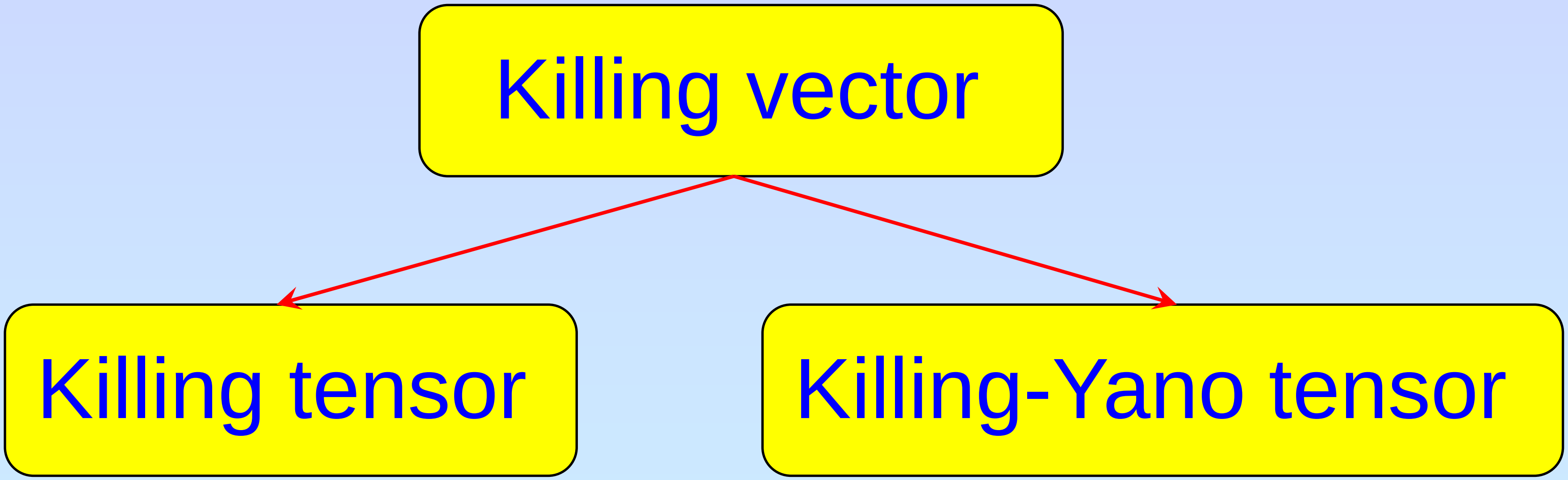
$$\xi_{(\mu;\nu)} = \tilde{\xi}\, g_{\mu\nu} \quad \text{(conformal Killing eq)}$$

$$\tilde{\xi} \equiv D^{-1} \xi^{\nu}{}_{;\nu}$$

D is # of ST dimensions

Hidden Symmetries

$$\xi_{(\mu;\nu)} = g_{\mu\nu}\,\tilde{\xi}$$



Symmetric generalization

CK=Conformal Killing tensor

$$K_{\mu_1\mu_2\ldots\mu_n} = K_{(\mu_1\mu_2\ldots\mu_n)}, \quad \tilde{K}_{\mu_2\ldots\mu_n} \sim \nabla^{\mu_1} K_{\mu_1\mu_2\ldots\mu_n}$$

$$K_{(\mu_1\mu_2\ldots\mu_n;\nu)} = g_{\nu(\mu_1} \tilde{K}_{\mu_2\mu_3\ldots\mu_n)}$$

Integral of motion

$$u^{\nu}(K_{\mu_1\mu_2\ldots\mu_n} u^{\mu_1} u^{\mu_2} \ldots u^{\mu_n})_{;\nu} = \varepsilon\, \tilde{K}_{\mu_1\mu_2\ldots\mu_{n-1}} u^{\mu_1} u^{\mu_2} \ldots u^{\mu_{n-1}}$$

Antisymmetric generalization

CY=Conformal Killing-Yano tensor

$$k_{\mu_1\mu_2\ldots\mu_n} = k_{[\mu_1\mu_2\ldots\mu_n]}, \quad \tilde{k}_{\mu_2\ldots\mu_n} \sim \nabla^{\mu_1} k_{\mu_1\mu_2\ldots\mu_n}$$

$$\nabla_{(\mu_1} k_{\mu_2)\mu_3\ldots\mu_{n+1}} = g_{\mu_1\mu_2} \tilde{k}_{\mu_3\ldots\mu_{n+1}} - (n-1) g_{[\mu_3(\mu_1} \tilde{k}_{\mu_2)\ldots\mu_{n+1}]}$$

If the rhs vanishes f=k is a Killing-Yano tensor

$K_{\mu\nu} = f_{\mu\mu_2 \dots \mu_n} f_{\nu}{}^{\mu_2 \dots \mu_n}$ is the Killing tensor

If $f_{\mu_1 \dots \mu_n}$ *is a KY tensor then for a geodesic motion the tensor* $p_{\mu_1 \dots \mu_{n-1}} = f_{\mu_1 \dots \mu_n} u^{\mu_n}$ *is parallelly propagated along a geodesic.*

Principal conformal Killing-Yano tensor

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \quad (*)$$

$$\nabla_{[a} h_{bc]} = 0, \quad \xi_a = \tfrac{1}{D-1}\nabla^n h_{na}$$

$$\nabla_X h = \frac{1}{D-1} X^\vee \wedge \delta h, \quad (*)$$

$$h = db, \qquad D = 2n + \varepsilon$$

PCKY tensor is a closed non-degenerate (matrix rank 2n) 2-form obeying (*)

Properties of CKY tensor

Hodge dual of CKY tensor is CKY tensor

Hodge dual of closed CKY tensor is KY tensor

External product of two closed CKY tensors is a closed CKY tensor

Darboux Basis

$$g_{ab} = \sum_{\mu} (e_a^{\mu} e_b^{\mu} + e_a^{\hat{\mu}} e_b^{\hat{\mu}}) + \varepsilon e_a^{n+1} e_b^{n+1},$$

$$h_{ab} = \sum_{\mu} x_{\mu} e_a^{\mu} \wedge e_b^{\hat{\mu}}$$

$$m_{\pm}^{\mu} = \frac{1}{\sqrt{2}} (e^{\hat{\mu}} \pm i e^{\mu})$$

Canonical Coordinates

$$h \bullet m_{\pm}^{\mu} = \mp i x^{\mu} m_{\pm}^{\mu}$$

A non-degenerate 2-form h has n independent eigenvalues

n essential coordinates x^{μ} and $n+\varepsilon$ Killing coordinates ψ_j are used as canonical coordinates

Principal conformal KY tensor

Non-degeneracy:

(1) Eigen-spaces of h are 2-dimensional

(2) x_μ are functionally independent in some domain

(they can be used as essential coordinates)

(1) is proved by Houri, Oota and Yasui e-print **arXiv:0805.3877**

(2) Case when some of eigenvalues are constant studied in
Houri, Oota and Yasui **Phys.Lett.B666:391-394,2008**.
e-Print: **arXiv:0805.0838**

Killing-Yano Tower

Killing-Yano Tower: Killing Tensors

$$h \Rightarrow \quad h^{\wedge 2} = h \wedge h \quad \Rightarrow \ldots \Rightarrow h^{\wedge j} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j\ times} \quad \Rightarrow h^{\wedge n} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{n\ times}$$

$2 \qquad 4 \qquad 2j \qquad 2n$

$$k_1 = *h \qquad k_2 = *h^{\wedge 2} \qquad k_j = *h^{\wedge j} \qquad k_n = *h^{\wedge n}$$

$D-2 \qquad D-4 \qquad D-2j \qquad D-2n = \varepsilon$

$$K^1 = k_1 \bullet k_1 \quad K^2 = k_2 \bullet k_2 \qquad K^j = k_j \bullet k_j \qquad K^n = k_n \bullet k_n$$

Set of (n-1) nontrivial rank 2 Killing tensors

$$\mathbf{K}_{ab}^{j} = \sum_{\mu=1}^{n} A_{\mu}^{j}(e_a^{\mu} e_b^{\mu} + e_a^{\hat{\mu}} e_b^{\hat{\mu}}) + \varepsilon A^{j} e_a^{2n+1} e_b^{2n+1}$$

$$A_{\mu}^{i} = \sum_{\substack{\nu_1,\ldots,\nu_i \\ \nu_1<\ldots<\nu_i \\ \nu_j \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_i}^2, \quad A^{i} = \sum_{\substack{\nu_1,\ldots,\nu_i \\ \nu_1<\ldots<\nu_i}} x_{\nu_1}^2 \ldots x_{\nu_i}^2$$

Killing-Yano Tower: Killing Vectors

$\xi_a = \frac{1}{D-1} \nabla^n h_{na}$ is a primary Killing vector

$\nabla_{(a} \xi_{b)} = \frac{1}{D-2} R_{n(a} h_{b)}{}^n$ (*)

On-shell ($R_{ab} \sim \Lambda g_{ab}$) (*) implies $\xi_{(a;b)} = 0$

Off-shell it is also true but the proof is much complicated (see Krtous, V.F., Kubiznak (2008))

$$\xi_1 = K_1 \bullet \xi \quad \Rightarrow \xi_2 = K_2 \bullet \xi \quad \Rightarrow \ldots \Rightarrow \xi_j = K_j \bullet \xi \quad \ldots \Rightarrow \xi_{n-1} = K_{n-1} \bullet \xi$$

Total number of the Killing vectors is $n + \varepsilon$

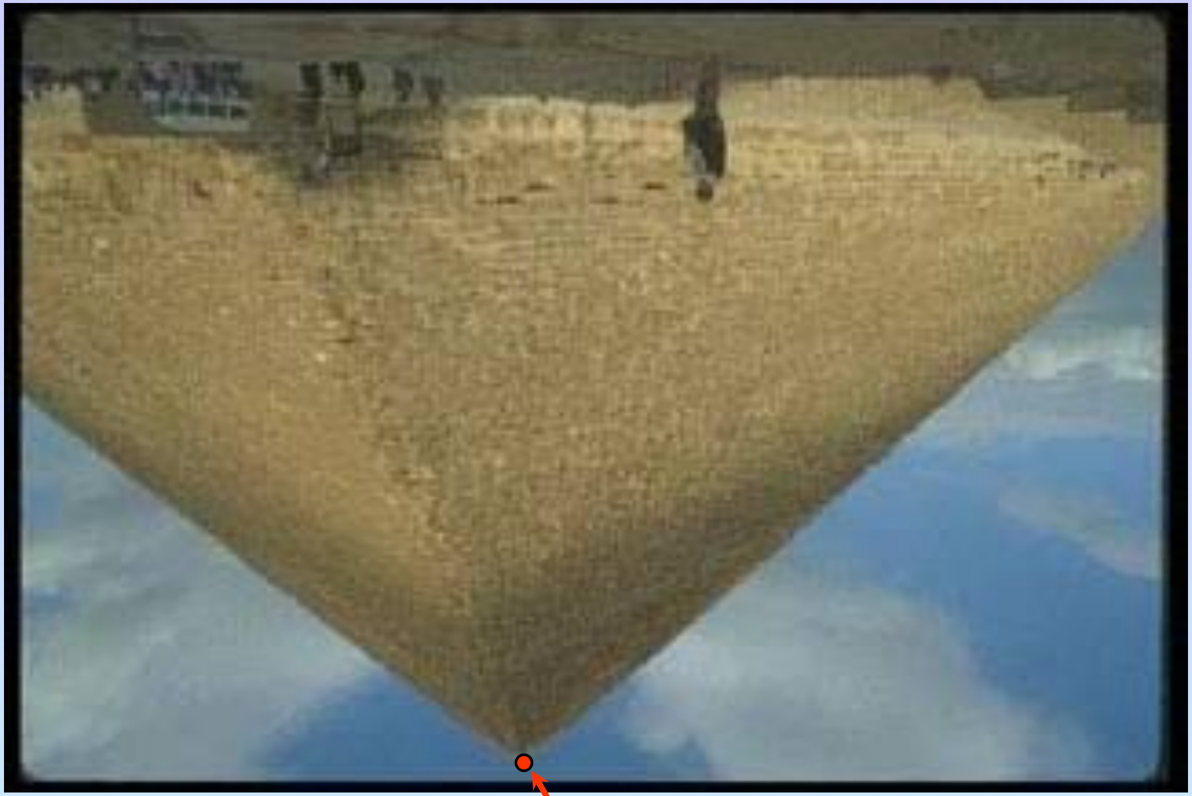
$$d\psi_i = \sum_\mu \frac{(-x_\mu^2)^{n-1-i}}{U_\mu \sqrt{Q_\mu}} e^{\hat{\mu}}, \quad \xi_i = \partial_{\psi_i}, \quad U_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} (x_\nu^2 - x_\mu^2)$$

Total number of conserved quantities

$$(n+\varepsilon) + (n-1) + 1 = 2n + \varepsilon = D$$

$KV \qquad KT \qquad g$

Reconstruction of metric

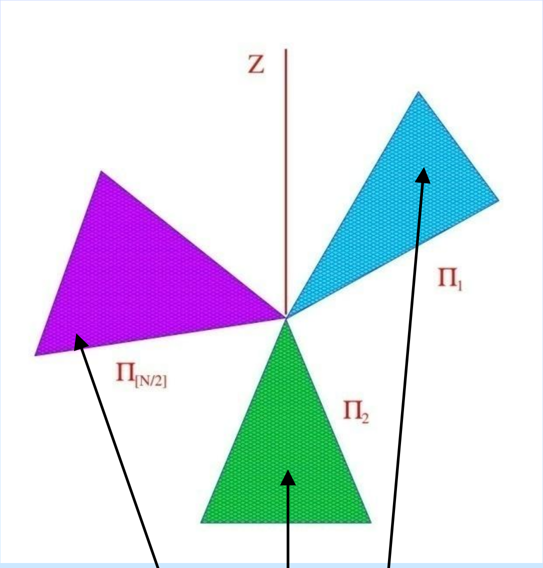


Principal Conformal Killing-Yano Tensor

Coordinates

Killing coordinates: $\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n$

Essential coordinates: $x_1, \ldots, x_n$



2-planes of rotation

Off-Shell Results

A metric of a spacetime which admits a (non-degenerate) principal CKY tensor can be written in the canonical form.

$$e^{\mu} = \frac{1}{\sqrt{Q_{\mu}}} dx_{\mu}, \quad e^{\hat{\mu}} = \sqrt{Q_{\mu}} \sum_{i=0}^{n-1} A_{\mu}^{i} \, d\psi_{i}$$

$$Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad X_{\mu} = X_{\mu}(x_{\mu})$$

Houri, Oota, and Yasui [PLB (2007); JP A41 (2008)] proved this result under additional assumptions: $L_{\xi} g = 0$ and $L_{\xi} h = 0$. Recently Krtous, V.F., Kubiznak [arXiv:0804.4705 (2008)] and Houri, Oota, and Yasui [arXiv:0805.3877 (2008)] proved this without additional assumptions.

On-Shell Result

A solution of the vacuum Einstein equations with the cosmological constant which admits a (non-degenerate) principal CKY tensor coincides with the Kerr-NUT-(A)dS spacetime.

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k}$$

Kerr-NUT-(A)dS spacetime is the most general BH solution obtained by Chen, Lu, and Pope [CQG (2006)]; See also Oota and Yasui [PL B659 (2008)]

"General Kerr-NUT-AdS metrics in all dimensions", Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$n = \left[D/2 \right], \;\; D = 2n + \varepsilon$$

$$R_{\mu\nu} = (D-1)\lambda g_{\mu\nu}$$

$$\lambda, M - mass, a_k - (n-1+\varepsilon)\ rotation\ \ parameters,$$

$$M_{\alpha} - (n-1-\varepsilon)\ `NUT'\ parameters$$

$$Total\ \#\ of\ \ parameters\ is\ D - \varepsilon$$

Illustrations

Let us consider a 4D flat spacetime with the metric

$$dS^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

and let b be the following 1-form

$$b = \tfrac{1}{2}[-R^2\, dT + a(Y\, dX - X\, dY)], \quad R^2 = X^2 + Y^2 + Z^2$$

One has

$$h = db = dT \wedge (XdX + YdY + ZdZ) + adY \wedge dX$$

$$f = {*h} = X\, dZ \wedge dY + Z\, dY \wedge dX + Y\, dX \wedge dZ + a\, dZ \wedge dT$$

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \qquad (*)$$

$$\nabla_{[a} h_{bc]} = 0, \qquad \xi_a = \tfrac{1}{D-1}\nabla^n h_{na}$$

$$h_{na} = \begin{pmatrix} 0 & X & Y & Z \\ -X & 0 & -a & 0 \\ -Y & a & 0 & 0 \\ -Z & 0 & 0 & 0 \end{pmatrix}$$

$$\xi_a^{(0)} = \tfrac{1}{3} h_{na}{}^{,n} = (-1, 0, 0, 0), \qquad \xi^{(0)a}\partial_a = \partial_T$$

$$h_{TX,X} = 1 = -g_{XX}\xi_T^{(0)}, \qquad etc.$$

$$K_{ab} = \begin{pmatrix} a^2 & -aY & aX & 0 \\ -aY & Y^2 + Z^2 & -XY & -XZ \\ aX & -XY & X^2 + Z^2 & -YZ \\ 0 & -XY & -YZ & X^2 + Y^2 - a^2 \end{pmatrix}$$

$$\mathrm{K} = K_{ab} p^a p^b = \vec{L}^2 + 2aEL_Z + a^2 (E^2 - p_Z^2)$$

$$\xi^{(\psi)} = -K^{ab} \xi_b^{(0)} \partial_a = a^2 \partial_T + a[Y \partial_X - X \partial_Y] = a^2 \partial_T - a \partial_\varphi$$

$$t = T + a\varphi, \quad \psi = -\varphi / a,$$

$$\partial_T = \partial_t, \quad \partial_\varphi = a \partial_t - a^{-1} \partial_\psi;$$

$$\xi^{(0)} = \partial_t, \quad \xi^{(\psi)} = \partial_\psi$$

Darboux coordinates

$$H^a{}_b = h^{ac} h_{bc}, \quad \Pi^a{}_b = H^a{}_b - \Lambda \delta^a{}_b;$$

$$\Pi = \begin{pmatrix} -R^2 - \Lambda & aY & -aX & 0 \\ -aY & a^2 - X^2 - \Lambda & -XY & -XZ \\ aX & -XY & a^2 - Y^2 - \Lambda & -YZ \\ 0 & -XZ & -YZ & -Z^2 - \Lambda \end{pmatrix}$$

$$\det(\Pi) = 0 \quad \Rightarrow \quad \Lambda^2 + (R^2 - a^2)\Lambda - a^2 Z^2 = 0,$$

$$\Lambda_\pm = \tfrac{1}{2}[a^2 - R^2 \pm \sqrt{(R^2 - a^2)^2 + 4a^2 Z^2}],$$

$r^2 = -\Lambda_-, \quad y^2 = \Lambda_+$ are essential coordinates

$$X = a^{-1}\sqrt{(r^2+a^2)(a^2-y^2)}\cos\varphi,$$

$$Y = a^{-1}\sqrt{(r^2+a^2)(a^2-y^2)}\sin\varphi,$$

$$Z = a^{-1}ry,$$

$$dS^2 = -dT^2 + (r^2+y^2)[\frac{dr^2}{r^2+a^2} + \frac{dy^2}{a^2-y^2}] + a^{-2}(r^2+a^2)(a^2-y^2)d\varphi^2$$

$$T = t + a^2\psi, \quad \varphi = -a\psi$$

$$(*) \quad dS^2 = \frac{1}{r^2+y^2}[-R(dt+y^2d\psi)^2 + Y(dt-r^2d\psi)^2] + (r^2+y^2)[\frac{dr^2}{R} + \frac{dy^2}{Y}],$$

$$R = r^2 + a^2, \quad Y = a^2 - y^2$$

This is a flat ST metric in the Darboux coordinates associated with the potential b.

$$b = \tfrac{1}{2}[(y^2 - r^2 - a^2)dt - r^2 y^2 d\psi]$$

Three important results can be proved:

(i) For arbitrary $R(r)$ and $Y(y)$ b is a potential generating a closed conformal Killing-Yano tensor for the metric (*);

(ii) (*) with arbitrary functions $R(r)$ and $Y(y)$ is the most general metric, which admits a closed conformal Killing-Yano tensor;

(iii) (*) with arbitrary functions $R(r)$ and $Y(y)$ belongs to Petrov type D

The Einstein equations, $R_{\mu\nu} = -3\lambda g_{\mu\nu}$, for these metrics are satisfied iff the following equation is valid

$$\frac{d^2R}{dr^2} + \frac{d^2Y}{dy^2} = 12\lambda(r^2 + y^2).$$

The most general solution of this equation can be written in the form

$$R = (r^2 + a^2)(1 + \lambda r^2) - 2Mr, \quad Y = (a^2 - y^2)(1 - \lambda y^2) + 2Ny$$

Remark: the tranformation $r \to ix, \quad M \to iM$ makes the expressions for the metric (*), potential b, and the solutions symmetric with respect to the map: $x \leftrightarrow y$

Principal CKY tensor in Kerr-NUT-(A)dS

V. F., D.Kubiznak, Phys.Rev.Lett. 98: 011101, 2007; gr-qc/0605058; D. Kubiznak, V. F., Class.Quant.Grav. 24: F1, 2007; gr-qc/0610144.

$$h = db, \quad b = \frac{1}{2}\sum_{k=0}^{n-1} A^{(k+1)}\, d\psi_k$$

(The same as in a flat ST in the Carter-type coordinates)

Complete integrability of geodesic motion in general Kerr-NUT-AdS spacetimes

D. N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, Phys.Rev.Lett. 98 :061102, 2007; hep-th/0611083

P. Krtous, D. Kubiznak, D. N. Page, V. F., JHEP 0702: 004, 2007; hep-th/0612029

Vanishing Poisson brackets for integrals of motion

Separability of Hamilton-Jacobi and Klein-Gordon equations in Kerr-NUT-(A)dS ST

V. F., P. Krtous , D. Kubiznak , JHEP (2007); hep-th/0611245;
Oota and Yasui, PL B659 (2008); Sergeev and Krtous, PRD 77 (2008).

Klein-Gordon equation

$$\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}g^{ab}\partial_b\Phi) = m^2\Phi.$$

Multiplicative separation

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu)\prod_{k=0}^{m} e^{i\Psi_k\psi_k}.$$

$$(X_\mu R'_\mu)' + \epsilon\frac{X_\mu}{x_\mu}R'_\mu - \frac{R_\mu}{X_\mu}\Big(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\Psi_k\Big)^2 - \sum_{k=0}^{m} b_k(-x_\mu^2)^{n-1-k}R_\mu = 0\,.$$

$$b_0 = m^2$$

Notes on Parallel Transport

Case 1: Parallel transport along timelike geodesics

Let u^a be a vector of velocity and h_{ab} be a PCKYT.

$P_a^b = \delta_a^b + u_a u^b$ is a projector to the plane orthogonal to u^a.

Denote $F_{ab} = P_a^c P_b^d h_{cd} = h_{ab} + u_a u^c h_{cb} + h_{ac} u^c u_b$

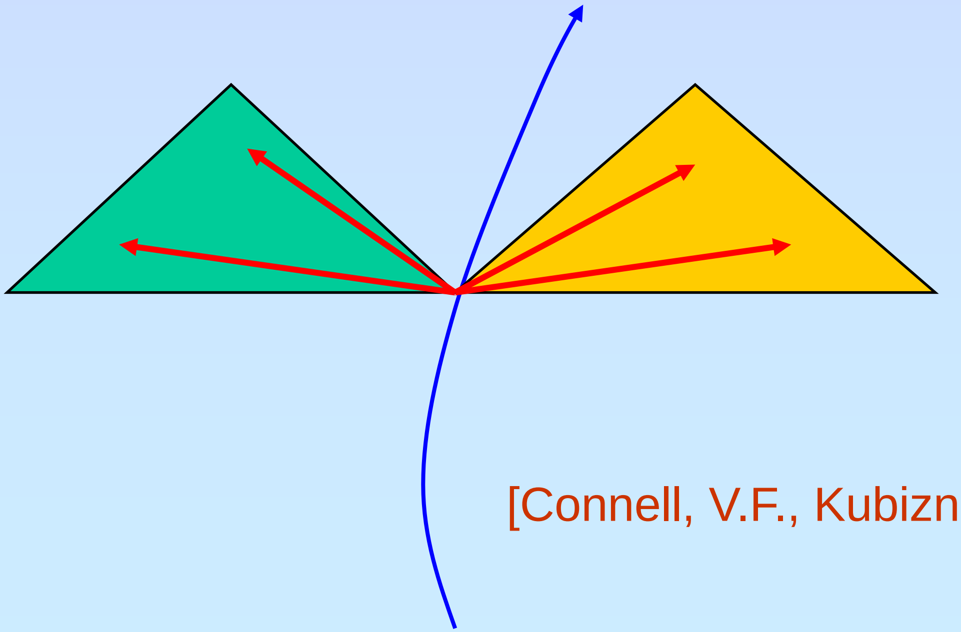
Lemma (Page): F_{ab} is parallel propagated along a geodesic:

$$\nabla_u F_{ab} = 0$$

Proof: We use the definition of the PCKYT

$$\nabla_u h_{ab} = u_a \xi_b - \xi_a u_b$$

Suppose h_{ab} is a non-degenerate, then for a generic geodesic eigen spaces of F_{ab} with non-vanishing eigen values are two dimensional. These 2D eigen spaces are parallel propagated. Thus a problem reduces to finding a parallel propagated basis in 2D spaces. They can be obtained from initially chosen basis by 2D rotations. The ODE for the angle of rotation can be solved by a separation of variables.



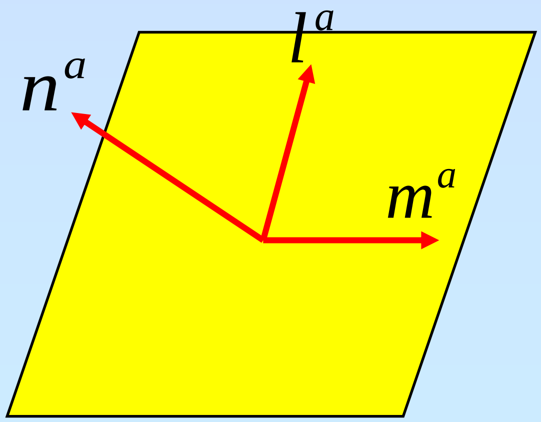
[Connell, V.F., Kubiznak, PRD 78, 024042 (2008)]

Case 2: Parallel transport along null geodesics

Let l^a be a tangent vector to a null geodesic and k^a be a parallel propagated vector obeying the condition $l^a k_a = 0.$

Then the vector $w^a = k_b h^{ba} + \beta l^a$ is parallel propagated, provided $\dot{\beta} = k_a \xi^a.$

This procedure allows one to construct 2 more parallel propagated vectors m^a and $n^a,$ starting with $l^a.$



We introduce a projector $P_{ab} = g_{ab} + 2l_{(a}n_{b)}$, and $F_{ab} = P_a^c P_b^d h_{cd}$.

One has: $\nabla_l F_{ab} = P_a^c P_b^d \nabla_l h_{cd} = 2P_a^c P_b^d l_{[c}\xi_{d]} = 0.$

Thus F_{ab} is parallel propaged along a null geodesic. We use rotations in its 2D eigen spaces to construct a parallel propagated basis.

[Kubiznak,V.F.,Krtous, Connell, PRD 79, 024018 (2009)]

More Recent Developments

Separability of the massive Dirac equation in the Kerr-NUT-(A)dS spacetime [Oota and Yasui, Phys. Lett. B 659, 688 (2008)]

Stationary string equations in the Kerr-NUT-(A)dS spacetime are completely integrable.
[D. Kubiznak, V. F., JHEP 0802:007,2008; arXiv:0711.2300]

Solving equations of the parallel transport along geodesics [P. Connell, V. F., D. Kubiznak, PRD,78, 024042 (2008); arXiv:0803.3259; D. Kubiznak, V. F., P. Connell, arXiv:0811.0012 (2008)]

Einstein spaces with degenerate closed conformal KY tensor [Houri, Oota and Yasui **Phys.Lett.B666:391-394,2008**. e-Print: **arXiv:0805.0838]**

Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime
[Oota, Yasui, arXiv:0812.1623]

On the supersymmetric limit of Kerr-NUT-AdS metrics [Kubiznak, arXiv:0902.1999]

GENERALIZED KILLING-YANO TENSORS

[Kubiznak, Kunduri, and Yasui, 0905.0722 (2009)]

Minimally gauged supergravity (5D EM with Chern-Simons term):

$$L = *(R+\Lambda) - \tfrac{1}{2} F \wedge *F + \tfrac{1}{3\sqrt{3}} F \wedge F \wedge A,$$

$$dF = 0, \quad d * F - \tfrac{1}{\sqrt{3}} F \wedge F = 0,$$

$$R_{ab} + \tfrac{1}{3}\Lambda g_{ab} = \tfrac{1}{2}\left(F_{ac}F_{b}{}^{c} - \tfrac{1}{6} g_{ab} F^2\right)$$

Torsion: $T = \frac{1}{\sqrt{3}} * F$

$$\nabla^T_c h_{ab} = \nabla_c h_{ab} - \tfrac{1}{\sqrt{3}} (*F)_{cd[a} h^d{}_{b]} = 2 g_{c[a}\xi_{b]},$$

$$K_{ab} = (*h)_{acd} (*h)_b{}^{cd} = h_{ac} h_b{}^c - \tfrac{1}{2} g_{ab} h^2$$

Application: Chong, Cvetic, Lu, Pope [PRL, 95,161301,2005]

Note: This is type I metric.

Summary

The most general spacetime admitting the PCKY tensor is described by the canonical metric. It has the following properties:

- It is of the algebraic type D
- It allows a separation of variables for the Hamilton-Jacoby, Klein-Gordon, Dirac and stationary string equations
- The geodesic motion in such a spacetime is completely integrable. The problem of finding parallel-propagated frames reduces to a set of the first order ODE
- When the Einstein equations with the cosmological constant are imposed the canonical metric becomes the Kerr-NUT-(A)dS spacetime